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E. Livne
Associate Editor

Characteristics of Sensitivity Analysis of Repeated Frequencies

D. Wang,* J. S. Jiang,† and W. H. Zhang‡
Northwestern Polytechnical University, 710072 Xi'an, People's Republic of China

I. Introduction

THE design sensitivity analysis of a natural frequency of a vibration system has become increasingly important for various gradient-based optimization algorithms. For instance, it is used for studying the effect of a design modification, finding the search direction, and constructing the frequency approximation. However, the occurrence of repeated frequencies constitutes one of the main difficulties in structural optimizations with dynamic frequency constraints because of nondifferentiability of a repeated frequency.[1] When repeated frequencies arise in a system, the related vibration mode shape can not be uniquely determined. Any linear combination of the modes is still valid for the repeated frequency. The present paper investigates the design sensitivity computation of a repeated frequency in the vibration mode space.

It is widely recognized that a repeated frequency is not generally differentiable in the common sense, that is, the Frechet derivative does not exist. Only directional derivatives can be obtained.[1] The nondifferentiability of a repeated frequency may be attributed to the fact that the mode corresponding to a repeated frequency has a great deal of uncertainty compared to a distinct one. Nonetheless, repeated frequencies are often involved in dynamic systems. They are associated with the structural symmetry or induced by the frequency evolution in the design optimization.[2] Haug et al.[1] and Choi et al.[3] demonstrated that a repeated frequency is directionally differentiable. Mills-Curran[4] found that the sensitivities of a repeated frequency can be evaluated by solving a subeigenvalue problem. Seyranian et al.[5] dealt with the frequency sensitivity by means of a design perturbation approach. Friswell[6] demonstrated that the first-order Taylor series of a repeated frequency is less valid for several design parameters. Sergeyev and Mroz[2] computed the design sensitivity along a special path in the design space. Pedersen and Nielsen[7] made an attempt to describe the sensitivity computation in the eigenspace. To avoid mode selection, Prells and Friswell[8] even presented an algorithm to calculate the sensitivity without explicit use of vibration modes.

The objective of this Note is to reveal and demonstrate some interesting properties for the design sensitivity of a repeated frequency. It will be shown that the sensitivity calculation can be obtained with additional physical interpretations in the vibration mode space. Furthermore, the characteristics of the sensitivity analysis will be investigated systematically. These characteristics will lay the foundations for the standard gradient-based optimization algorithms. In accordance with these characteristics, the maximum of the fundamental frequency can be simply determined. By analogy, the proposed approach can be extended straightforwardly to the sensitivity analysis of repeated buckling load factors.

II. Sensitivity Analysis of a Natural Frequency

When a dynamic problem is considered, the eigenvalue equation is given as

$$\left([K]-\omega_j^2[M]\right)\{\phi\}_j=\{0\},\qquad j=1,\ldots,n \tag{1}$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices, respectively, which are assumed real, symmetric, and differentiable; ω_j is the jth natural frequency, $\{\phi\}_j$ is the corresponding vibration mode, and n is the number of degrees of freedom. The frequencies are real and ordered in increasing magnitude:

$$0\le\omega_1^2\le\omega_2^2\le\cdots\le\omega_n^2 \tag{2}$$

All of the modes are $[M]$ orthonormalized so that

$$\{\phi\}_i^T[K]\{\phi\}_j=\omega_j^2\delta_{ij}$$
$$\{\phi\}_i^T[M]\{\phi\}_j=\delta_{ij},\qquad i,j=1,\ldots,n \tag{3}$$

where δ_{ij} is the Kronecker delta.

A. Distinct Frequency Sensitivity

Suppose we have a distinct eigenpair $(\omega_j, \{\phi\}_j)$, the first-order derivative of the frequency with respect to a design parameter x is computed as[4]

$$\frac{\partial\omega_j^2}{\partial x}=\{\phi\}_j^T\left(\frac{\partial[K]}{\partial x}-\omega_j^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_j,\qquad j=1,\ldots,n \tag{4}$$

In most cases, Eq. (4) can be simplified at the element level,

$$\frac{\partial\omega_j^2}{\partial x}=\sum_{e=1}^{n_k}\{\phi_e\}_j^T\left(\frac{\partial[k_e]}{\partial x}-\omega_j^2\frac{\partial[m_e]}{\partial x}\right)\{\phi_e\}_j,\qquad j=1,\ldots,n \tag{5}$$

where $[k_e]$ and $[m_e]$ are the element stiffness and mass matrices, respectively; $\{\phi_e\}_j$ is the jth mode of the eth element, which contains only the related components of $\{\phi\}_j$; and n_k is the number of elements related to the design parameter. Because of the uniqueness of the mode shape, the design sensitivity can be calculated definitely and explicitly.

B. Repeated Frequency Sensitivity

For simplicity of presentation, we shall only treat a double repeated frequency with the proposed methodology. The derivation for higher-order multiplicity can be performed in a similar way. Assume that a double repeated frequency and its corresponding modes are

$$\left(\omega^2,\{\phi\}_1,\{\phi\}_2\right) \tag{6}$$

No special restrictions are imposed on the mode selections except the $[M]$ orthonormalization in compliance with Eq. (3). Therefore, as is well known, $\{\phi\}_1$ and $\{\phi\}_2$ can span a two-dimensional eigenspace, which is a subspace of the structural mode space. Any

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*Lecturer, Department of Aeronautical Structural Engineering, P.O. Box 118; wangdng66@yahoo.com.

†Professor, Institute of Vibration Engineering.

‡Professor, Department of Aircraft Manufacturing Engineering.

vector in this eigenspace, which is a linear combination of the two basis modes $\{\phi\}_1$ and $\{\phi\}_2$, is also a valid mode to the repeated frequency. That is,

$$\{\tilde{\phi}\} = c_1\{\phi\}_1 + c_2\{\phi\}_2 \tag{7}$$

$$\{\tilde{\phi}\}^T[M]\{\tilde{\phi}\} = 1 \Rightarrow c_1^2 + c_2^2 = 1 \tag{8}$$

where c_1 and c_2 are two arbitrary constants. In fact, Eqs. (7) and (8) can only determine the norm of $\{\tilde{\phi}\}$. The orientation or direction of $\{\tilde{\phi}\}$, or the ratio of c_1 to c_2, remains unknown. By substituting $\{\tilde{\phi}\}$ of Eq. (7) into Eq. (4), we can get the directional derivative associated with a particular mode $\{\tilde{\phi}\}$ in the vibration mode space:

$$\frac{\partial\omega^2}{\partial x} = c_1^2 g_{11} + c_2^2 g_{22} + 2c_1c_2g_{12} = \{C\}^T[G]\{C\} \tag{9}$$

in which

$$g_{mn} = \{\phi\}_m^T\left(\frac{\partial[K]}{\partial x} - \omega^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_n, \qquad m, n = 1, 2$$

$$\{C\} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}, \qquad [G] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{10}$$

To emphasize its dependence on the vibration mode selection and to distinguish it from the common definition of the directional derivative in the parameter space, we will, in this context, refer to the directional derivative of a repeated frequency obtained by Eq. (9) as the eigenspace directional derivative. In Eq. (10), g_{11} and g_{22} are defined as the basic eigenspace directional derivatives associated with two specific bases $\{\phi\}_1$ and $\{\phi\}_2$, respectively, and $g_{12} = g_{21}$ is the mixed eigenspace directional derivative. Then, $[G]$ is defined as the eigenspace directional derivative matrix related to two linearly independent basis modes. Here, we do not need the off-diagonal term g_{12} or g_{21} to vanish by any means.

To find the extreme values of $\partial\omega^2/\partial x$, we formulate the Lagrangian function with Eqs. (8) and (9):

$$L(c_1, c_2, \lambda) = \frac{\partial\omega^2}{\partial x} + \lambda\left(1 - c_1^2 - c_2^2\right)$$

$$= \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}^T \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} + \lambda\left(1 - c_1^2 - c_2^2\right) \tag{11}$$

where λ is the Lagrange multiplier. By differentiating the preceding expression with respect to the two parameters c_1 and c_2, respectively, and setting them to zeros, one gets

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \lambda \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \tag{12}$$

Clearly, this is an algebraic eigenvalue problem. We denote the eigenpairs of matrix $[G]$ as

$$(g_a, \{C\}_a), \qquad (g_b, \{C\}_b) \tag{13}$$

The following relations hold:

$$g_a \le g_b, \qquad \{C\}_a^T \cdot \{C\}_b = 0 \tag{14}$$

Therefore, the two stationary values (or extrema) of the eigenspace directional derivative computed in Eq. (9) are

$$\left.\frac{\partial\omega^2}{\partial x}\right|_{\text{sta1}} = \{C\}_a^T \cdot [G] \cdot \{C\}_a = g_a \tag{15a}$$

$$\left.\frac{\partial\omega^2}{\partial x}\right|_{\text{sta2}} = \{C\}_b^T \cdot [G] \cdot \{C\}_b = g_b \tag{15b}$$

When constants of the vectors $\{C\}_a$ or $\{C\}_b$ are taken in Eq. (7), the associated vibration modes corresponding to the stationary values are determined as, respectively,

$$\{\tilde{\phi}\}_a = c_{1a}\{\phi\}_1 + c_{2a}\{\phi\}_2, \qquad \{\tilde{\phi}\}_b = c_{1b}\{\phi\}_1 + c_{2b}\{\phi\}_2 \tag{16}$$

Thus far, all of the problems with the eigenspace directional derivative seem to be solved.

III. Characteristics of Sensitivity Analysis of a Repeated Frequency

The subeigenvalue problem in Eq. (12) has been established by several researchers for the sensitivity calculation of a repeated frequency,[4] which means that the stationary values of the eigenspace directional derivative are just the sensitivities of a repeated frequency. Before we apply the results in practice, however, it is observed from Eq. (12) that the matrix $[G]$ depends not only on the design parameter, but also on the choice of basis modes $\{\phi\}_1$ and $\{\phi\}_2$. Therefore, it may be questioned whether the resulting solutions in Eqs. (15) and (16) are irrelevant to the choice of the basis modes. If the solutions are not irrelevant, the results would be almost meaningless. Thus far, an explicit demonstration of the irrelevance of the mode choice is not yet available in the literature.

Subsequently, the characteristics of the sensitivity analysis of a repeated frequency are studied and demonstrated systematically. For clarity of description, it is presumed that a double repeated frequency occurs with two distinct frequencies $\omega_1^2(x)$ and $\omega_2^2(x)$ coalescing at $x = x_c$. That is,

$$\left(\omega_1^2, \{\phi\}_1\right), \qquad \left(\omega_2^2, \{\phi\}_2\right)$$

$$\omega^2 = \omega_1^2(x_c) = \omega_2^2(x_c) \tag{17}$$

Feature 1: $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$ are $[M]$ orthonormal.

As already derived, $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$ are the modes related to the stationary values of the eigenspace directional derivative, respectively. Then,

$$\{\tilde{\phi}\}_a^T[M]\{\tilde{\phi}\}_b = \left(c_{1a}\{\phi\}_1^T + c_{2a}\{\phi\}_2^T\right)[M](c_{1b}\{\phi\}_1 + c_{2b}\{\phi\}_2)$$

$$= (c_{1a}\cdot c_{1b} + c_{2a}\cdot c_{2b}) = \{C\}_a^T \cdot \{C\}_b = 0 \tag{18}$$

Feature 2: g_a and g_b are independent of the choice of basis modes.

Now we construct a new eigenspace directional derivative matrix $[H]$ by choosing another pair of basis modes $\{\bar{\phi}\}_1$ and $\{\bar{\phi}\}_2$ corresponding to the repeated frequency. Doubtlessly, the new basis modes can be linearly combined by $\{\phi\}_1$ and $\{\phi\}_2$ as

$$\{\bar{\phi}\}_1 = [\{\phi\}_1, \{\phi\}_2]\begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}_1, \qquad \{\bar{\phi}\}_2 = [\{\phi\}_1, \{\phi\}_2]\begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}_2 \tag{19}$$

The requirement of $[M]$ orthonormalization for $\{\bar{\phi}\}_1$ and $\{\bar{\phi}\}_2$ yields

$$\{\bar{\phi}\}_1^T[M]\{\bar{\phi}\}_2 = 0 \Rightarrow \{\beta\}_1^T \cdot \{\beta\}_2 = 0$$

$$\{\bar{\phi}\}_m^T[M]\{\bar{\phi}\}_m = 1 \Rightarrow \{\beta\}_m^T \cdot \{\beta\}_m = 1 \qquad m = 1, 2 \tag{20}$$

From Eq. (7), a linear combination of $\{\bar{\phi}\}_1$ and $\{\bar{\phi}\}_2$ leads to

$$\{\widetilde{\phi}\} = \bar{c}_1\{\bar{\phi}\}_1 + \bar{c}_2\{\bar{\phi}\}_2 = [\{\bar{\phi}\}_1, \{\bar{\phi}\}_2]\begin{Bmatrix} \bar{c}_1 \\ \bar{c}_2 \end{Bmatrix}$$

$$= [\{\phi\}_1, \{\phi\}_2] \cdot [\{\beta\}_1, \{\beta\}_2] \cdot \{\bar{C}\} \tag{21}$$

In the same way as Eq. (9), the eigenspace directional derivative related to $\{\tilde{\phi}\}$ is evaluated:

$$\frac{\partial\omega^2}{\partial x} = \{\bar{C}\}^T[H]\{\bar{C}\} \tag{22}$$

where $[H]$ is formulated according to Eq. (10):

$$[H] = [\{\beta\}_1, \{\beta\}_2]^T \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} [\{\beta\}_1, \{\beta\}_2] \tag{23}$$

Regarding Eq. (20), we recognize that $[\{\beta\}_1, \{\beta\}_2]$ is an orthogonal matrix. That is, $[H]$ is an orthogonal transformation from matrix $[G]$. Then, $[H]$ has the same eigenvalues and trace as $[G]$

$$h_a = g_a, \qquad h_b = g_b \tag{24a}$$

$$\mathrm{tr}(H) = \mathrm{tr}(G) = g_{11} + g_{22} = \mathrm{const} \tag{24b}$$

Equation (24a) indicates that the eigenspace directional derivative matrix $[G]$ has the same eigenvalues with respect to any pair of the basis modes. Equation (24b) indicates that the sum of all of the basic eigenspace directional derivatives is a constant.

Now, this feature answers the question that arose at the beginning of this section. We have demonstrated that g_a and g_b are irrelevant to the selection of the basis modes and they are only dependent on the design parameter involved in $[G]$. Therefore, g_a and g_b do make physical sense and have definite values for a repeated frequency. They are just the directional derivatives of a repeated frequency with respect to the design parameter variation in the $-$ sign and the $+$ sign, respectively.[1]

Feature 3: $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$ are independent of the choice of basis modes, and moreover, the associated mixed eigenspace directional derivative g_{ab} is equal to zero.

From the properties of an orthogonal transformation of a matrix in Eq. (23), we have the following relationship about the eigenvectors between matrices $[H]$ and $[G]$:

$$[\{\bar{C}\}_a, \{\bar{C}\}_b]_H = [\{\beta\}_1, \{\beta\}_2]^{-1} \cdot [\{C\}_a, \{C\}_b]_G \tag{25}$$

Based on Eqs. (16) and (21), one obtains

$$\begin{aligned}[\{\tilde{\phi}\}_a, \{\tilde{\phi}\}_b]_H &= [\{\bar{\phi}\}_1, \{\bar{\phi}\}_2] \cdot [\{\bar{C}\}_a, \{\bar{C}\}_b]_H \\ &= [\{\phi\}_1, \{\phi\}_2] \cdot [\{\beta\}_1, \{\beta\}_2] \cdot [\{\beta\}_1, \{\beta\}_2]^{-1} \cdot [\{C\}_a, \{C\}_b]_G \\ &= [\{\phi\}_1, \{\phi\}_2] \cdot [\{C\}_a, \{C\}_b]_G = [\{\tilde{\phi}\}_a, \{\tilde{\phi}\}_b]_G \end{aligned} \tag{26}$$

Thus, it can be concluded that $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$ are independent of the selection of basis modes. Furthermore, we know

$$\begin{aligned} g_{ab} &= \{\tilde{\phi}\}_a^T \left(\frac{\partial [K]}{\partial x} - \omega^2 \frac{\partial [M]}{\partial x} \right) \{\tilde{\phi}\}_b \\ &= \{C\}_a^T [\{\phi\}_1, \{\phi\}_2]^T \left(\frac{\partial [K]}{\partial x} - \omega^2 \frac{\partial [M]}{\partial x} \right) [\{\phi\}_1, \{\phi\}_2] \{C\}_b \\ &= \{C\}_a^T [G] \{C\}_b = 0 \end{aligned} \tag{27}$$

Equation (27) indicates that $[G]$ is a diagonal matrix with respect to the basis modes $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$. Moreover, in such a case the eigenvalues of $[G]$ are the diagonal terms. This feature underlines that there always exists a set of orthonormal vibration modes to which the basic eigenspace directional derivatives are exactly the sensitivities of a repeated frequency. Or in other words, the sensitivities of a repeated frequency could be computed in the same way as that for a distinct frequency by using $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$, respectively. Therefore, we define $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$ as the primary modes of a repeated frequency associated with a design parameter. Ojalvo[9] utilized this set of modes to calculate the mode derivatives.

Feature 4: the eigenspace directional derivative $\partial\omega^2/\partial x$ is limited, and moreover,

$$\left.\frac{\partial \omega^2}{\partial x}\right|_{\min} = g_a, \qquad \left.\frac{\partial \omega^2}{\partial x}\right|_{\max} = g_b$$

According to the derivation of Eq. (15), we know that g_a and g_b are the stationary values of the eigenspace directional derivative of a repeated frequency. Moreover, it can be shown from Eq. (9) that

$$\begin{aligned} \left|\frac{\partial \omega^2}{\partial x}\right| &\le \left|c_1^2 g_{11}\right| + \left|c_2^2 g_{22}\right| + 2|c_1 c_2 g_{12}| \\ &< |g_{11}| + |g_{22}| + 2|g_{12}| < +\infty \end{aligned} \tag{28}$$

Because of the limited value of $\partial\omega^2/\partial x$, we can conclude that, at $x = x_c$,

$$\left.\frac{\partial \omega^2}{\partial x}\right|_{\min} = g_a, \qquad \left.\frac{\partial \omega^2}{\partial x}\right|_{\max} = g_b \tag{29}$$

This feature indicates that the sensitivities of a repeated frequency are the extreme values of the eigenspace directional derivative.

Feature 5: if $g_a = g_b$, then $\omega_1^2(x)$, and $\omega_2^2(x)$ will vary identically to the design variation at the coalescent point x_c.

Here $g_a = g_b$ means that the sensitivities of a repeated frequency are equal. From feature 4 we know that, in this case, the eigenspace directional derivative $\partial\omega^2/\partial x$ is of a unique value and is independent of the mode at the coalescent point, that is,

$$\left.\frac{\partial \omega^2}{\partial x}\right|_{x_c} = \mathrm{const} \tag{30}$$

Therefore, $\omega_1^2(x)$ and $\omega_2^2(x)$ will vary identically to the design parameter variation.

On the other hand, if all of the eigenspace directional derivatives of a repeated frequency are equal to any related mode, it is certain that the repeated frequency has the same sensitivities. Physically, this situation corresponds to the repeated frequencies of a symmetric structure as the repeated frequencies remain with regard to the variation of the design parameter.[6]

Note that, when $[G]$ has a repeated eigenvalue, $\{\tilde{\phi}\}_a$ and $\{\tilde{\phi}\}_b$ can not be computed uniquely because any linear combination of $\{C\}_a$ and $\{C\}_b$ is also an eigenvector of $[G]$ in Eq. (12). In fact, any pair of the orthonormal mode is the primary ones corresponding to the design parameter. Therefore, it is fairly simple to calculate the sensitivities of repeated frequencies in this particular case.

Feature 6: if $g_a \ne g_b$ and $g_a \cdot g_b \ge 0$, then $\omega_1^2(x)$ and $\omega_2^2(x)$ will change with similar tendency and their orders swap due to the design variation at the coalescent point x_c.

From Eqs. (9) and (10) we recognize that in this case, $\partial\omega^2/\partial x$ is relevant to the mode selection. However, because of $g_a \cdot g_b \ge 0$, the sign of the eigenspace directional derivatives remains unchanged to any mode, that is,

$$\mathrm{sign}\left(\left.\frac{\partial \omega^2}{\partial x}\right|_{x_c} \right) = \mathrm{const} \tag{31}$$

where $\mathrm{sign}(\cdot)$ is the sign function. Hence, the sensitivities of the repeated frequency are, respectively,

$$\left.\frac{\partial \omega_1^2}{\partial x}\right|_{x_c} = g_a, \qquad \left.\frac{\partial \omega_2^2}{\partial x}\right|_{x_c} = g_b \tag{32}$$

Therefore, $\omega_1^2(x)$ and $\omega_2^2(x)$ will increase or decrease simultaneously and then separate with the design variation. Their corresponding orders swap with each other in accordance with the conventional frequency-order definition.

Feature 7: if $g_a \ne g_b$ and $g_a \cdot g_b < 0$, then $\omega_1^2(x)$ and $\omega_2^2(x)$ will change oppositely at the coalescent point. Moreover, $\omega_1^2(x)$ reaches its maximum.

The inequality condition $g_a \cdot g_b < 0$ means that the sign of the sensitivities of $\omega_1^2(x)$ and $\omega_2^2(x)$ are different at $x = x_c$. As is derived earlier, in this case the sensitivities are, respectively,

$$\left.\frac{\partial \omega_1^2}{\partial x}\right|_{x_c} = g_a < 0, \qquad \left.\frac{\partial \omega_2^2}{\partial x}\right|_{x_c} = g_b > 0 \tag{33}$$

Therefore, $\omega_1^2(x)$ and $\omega_2^2(x)$ would change oppositely, that is, one increases while another decreases with the design parameter variation. In structural dynamics, we always assume $\omega_1 \le \omega_2$. Therefore, $g_a \cdot g_b < 0$ is a sufficient condition in which $\omega_1^2(x)$ takes its maximal value.

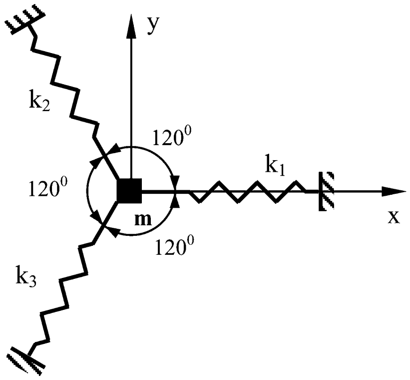

Fig. 1 Schematic planar spring–mass system.

IV. Illustrative Example

A schematic planar spring–mass system[10] is shown in Fig. 1. The natural frequencies may be changed by spring stiffness variations. The structural stiffness and mass matrices are, respectively,

$$[K] = \begin{bmatrix} k_1 + (1/4)(k_2 + k_3), & (\sqrt{3}/4)(k_3 - k_2) \\ (\sqrt{3}/4)(k_3 - k_2), & (3/4)(k_3 + k_2) \end{bmatrix}$$

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

The natural frequencies are

$$\omega^2 = (1/2m)\Big[(k_1 + k_2 + k_3) \pm \sqrt{(k_1 + k_2 + k_3)^2 - 3(k_1k_2 + k_2k_3 + k_1k_3)}\Big]$$

Assume $k_1 = k_2 = k_3 = k$: A double repeated frequency arises

$$\bar{\omega}_1^2 = \bar{\omega}_2^2 = \frac{3k}{2m}, \qquad [\Phi] = \frac{1}{\sqrt{m}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Three cases of design variations are taken into account:

Case 1, $k_1 = k + \delta k_1$, $k_2 = k_3 = k$, and assume $\delta k_1 > 0$, then one obtains

$$\omega_1^2 = \frac{3k}{2m}, \qquad \phi_1 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = \frac{3k + 2\delta k_1}{2m}, \qquad \phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

Case 2, $k_1 = k_2 = k$, $k_3 = k + \delta k_3$, and assume $\delta k_3 > 0$, then

$$\omega_1^2 = \frac{3k}{2m}, \qquad \phi_1 = \frac{1}{\sqrt{m}}\begin{Bmatrix} \sqrt{3}/2 \\ -1/2 \end{Bmatrix}$$

$$\omega_2^2 = \frac{3k + 2\delta k_3}{2m}, \qquad \phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 1/2 \\ \sqrt{3}/2 \end{Bmatrix}$$

Case 3, $k_1 = k + \delta k_1$, $k_2 = k$, $k_3 = k - \delta k_1$, and assume $\delta k_1 > 0$, then

$$\omega_1^2 = \frac{3k - \sqrt{3}\delta k_1}{2m}, \qquad \phi_1 = \frac{1}{\sqrt{m}}\begin{Bmatrix} (\sqrt{6} - \sqrt{2})/4 \\ (\sqrt{6} + \sqrt{2})/4 \end{Bmatrix}$$

$$\omega_2^2 = \frac{3k + \sqrt{3}\delta k_1}{2m}, \qquad \phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} (\sqrt{6} + \sqrt{2})/4 \\ -(\sqrt{6} - \sqrt{2})/4 \end{Bmatrix}$$

As can be seen, the vibration modes change discontinuously with design variations. Now let us apply sensitivity analysis of a repeated frequency to obtain the frequencies and modes of the perturbed structure.

First, when $k_1 \Rightarrow k + \delta k_1$ and others remain constant,

$$\frac{\partial[K]}{\partial k_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \frac{\partial[M]}{\partial k_1} = [0]$$

We select two basis modes for subsequent analyses:

$$\phi_1 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \qquad \phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Then, the eigenspace directional derivative matrix is formulated:

$$[G] = \frac{1}{m}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The eigenpairs of $[G]$ are, respectively,

$$g_a = 0, \qquad C_a = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}; \qquad g_b = \frac{1}{m}, \qquad C_b = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

The natural frequencies of the perturbed structure are calculated as follows:

$$\omega_1^2 = \bar{\omega}_1^2 + \frac{\partial \omega_1^2}{\partial k_1} \cdot \delta k_1 = \bar{\omega}_1^2 + g_a \cdot \delta k_1 = \frac{3k}{2m}$$

$$\omega_2^2 = \bar{\omega}_2^2 + \frac{\partial \omega_2^2}{\partial k_1} \cdot \delta k_1 = \bar{\omega}_2^2 + g_b \cdot \delta k_1 = \frac{3k + 2\delta k_1}{2m}$$

The primary vibration modes are, respectively,

$$\{\tilde{\phi}\}_a = c_{1a}\phi_1 + c_{2a}\phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\{\tilde{\phi}\}_b = c_{1b}\phi_1 + c_{2b}\phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

These results are identical to the dynamic analysis.

Second, when $k_3 \Rightarrow k + \delta k_3$ and others remain constant,

$$\frac{\partial[K]}{\partial k_3} = \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}, \qquad \frac{\partial[M]}{\partial k_3} = [0]$$

Then

$$[G] = \frac{1}{m}\begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}$$

The eigenpairs of $[G]$ are, respectively,

$$g_a = 0, \qquad C_a = \begin{Bmatrix} \sqrt{3}/2 \\ -1/2 \end{Bmatrix}; \qquad g_b = \frac{1}{m}, \qquad C_b = \begin{Bmatrix} 1/2 \\ \sqrt{3}/2 \end{Bmatrix}$$

The natural frequencies of the perturbed structure are

$$\omega_1^2 = \bar{\omega}_1^2 + \frac{\partial \omega_1^2}{\partial k_3} \cdot \delta k_3 = \bar{\omega}_1^2 + g_a \cdot \delta k_3 = \frac{3k}{2m}$$

$$\omega_2^2 = \bar{\omega}_2^2 + \frac{\partial \omega_2^2}{\partial k_3} \cdot \delta k_3 = \bar{\omega}_2^2 + g_b \cdot \delta k_3 = \frac{3k + 2\delta k_3}{2m}$$

The primary modes are, respectively,

$$\{\tilde{\phi}\}_a = \frac{1}{\sqrt{m}}\begin{Bmatrix} \sqrt{3}/2 \\ -1/2 \end{Bmatrix}, \qquad \{\tilde{\phi}\}_b = \frac{1}{\sqrt{m}}\begin{Bmatrix} 1/2 \\ \sqrt{3}/2 \end{Bmatrix}$$

The results are again identical to the dynamic analysis for the perturbed system. Also, $\{\tilde{\phi}\}_b$ is along the direction of the spring k_3.

Note that if the spring stiffness increment is negative, the order of the modes has to be switched in accordance with the frequency order definition in dynamics.

Third, more than one spring stiffness is changed in this case, that is, $k_1 \Rightarrow k + \delta k_1$ and $k_3 \Rightarrow k - \delta k_1$. The design variation is $\boldsymbol{s} = (1, 0, -1)\delta k_1$.

Approach 1: Let us treat this case with the method of the directional derivative in the design parameter space,

$$\frac{\mathrm{d}[K]}{\mathrm{d}\boldsymbol{s}} = \frac{\partial [K]}{\partial k_1} - \frac{\partial [K]}{\partial k_3} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 \end{bmatrix}$$

$$\frac{\mathrm{d}[M]}{\mathrm{d}\boldsymbol{s}} = [0]$$

Thus,

$$[G] = \frac{1}{m} \begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 \end{bmatrix}$$

The eigenpairs of $[G]$ are

$$g_a = -\frac{\sqrt{3}}{2m}, \qquad C_a = \begin{Bmatrix} 1/\left(\sqrt{6} + \sqrt{2}\right) \\ 1/\left(\sqrt{6} - \sqrt{2}\right) \end{Bmatrix}$$

$$g_b = \frac{\sqrt{3}}{2m}, \qquad C_b = \begin{Bmatrix} 1/\left(\sqrt{6} - \sqrt{2}\right) \\ -1/\left(\sqrt{6} + \sqrt{2}\right) \end{Bmatrix}$$

Therefore, the sensitivities along the specified direction of the design variation are g_a and g_b, respectively. The natural frequencies of the perturbed structure are calculated as

$$\omega_1^2 = \bar{\omega}_1^2 + \frac{\mathrm{d}\omega_1^2}{\mathrm{d}\boldsymbol{s}} \cdot \delta k_1 = \bar{\omega}_1^2 + g_a \cdot \delta k_1 = \frac{3k - \sqrt{3}\delta k_1}{2m}$$

$$\omega_2^2 = \bar{\omega}_2^2 + \frac{\mathrm{d}\omega_2^2}{\mathrm{d}\boldsymbol{s}} \cdot \delta k_1 = \bar{\omega}_2^2 + g_b \cdot \delta k_1 = \frac{3k + \sqrt{3}\delta k_1}{2m}$$

Because $g_a \cdot g_b < 0$, we can conclude that, in the design variation direction $\boldsymbol{s} = (1, 0, -1)$, the fundamental frequency reaches its maximum:

$$\omega_1^2\big|_{\max} = 3k/2m$$

Approach 2: For the last case, we can get the results by using the primary modes obtained earlier. From the first two cases, we know that the primary modes related to $k_1 \Rightarrow k + \delta k_1$ are

$$\tilde{\Phi}_1 = [\{\tilde{\phi}\}_a, \{\tilde{\phi}\}_b]_1 = \frac{1}{\sqrt{m}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The primary modes related to $k_3 \Rightarrow k - \delta k_1$ are

$$\tilde{\Phi}_3 = [\{\tilde{\phi}\}_a, \{\tilde{\phi}\}_b]_3 = \frac{1}{\sqrt{m}} \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

It is clear that the primary modes corresponding to each design variation are not coincident with each other. Therefore, the frequency can not be approximated by a Taylor series in the two design variations.[6] That is, it is inappropriate or less valid to calculate the natural frequencies by

$$\omega^2 \neq \bar{\omega}^2 + \frac{\partial \omega^2}{\partial k_1} \cdot \delta k_1 + \frac{\partial \omega^2}{\partial k_3} \cdot \delta k_3$$

However, we can first construct the corresponding primary modes. Because of the equal perturbed magnitude of the spring stiffnesses, a pair of orthogonal vibration modes is obtained simply:

$$\Phi = \tilde{\Phi}_1 + \tilde{\Phi}_3 = \frac{1}{\sqrt{m}} \begin{bmatrix} 1/2 & \left(2 + \sqrt{3}\right)/2 \\ \left(2 + \sqrt{3}\right)/2 & -1/2 \end{bmatrix}$$

By normalizing Φ, one can get the primary modes of the perturbed system:

$$\tilde{\Phi} = \frac{\sqrt{2}}{\sqrt{3} + 1}\Phi = \frac{1}{\sqrt{m}} \begin{bmatrix} \left(\sqrt{6} - \sqrt{2}\right)/4 & \left(\sqrt{6} + \sqrt{2}\right)/4 \\ \left(\sqrt{6} + \sqrt{2}\right)/4 & -\left(\sqrt{6} - \sqrt{2}\right)/4 \end{bmatrix}$$

Therefore, the natural frequencies of the perturbed system are calculated as

$$\mathrm{diag}\left(\omega_1^2, \omega_2^2\right) = \tilde{\Phi}^T [K] \tilde{\Phi} = \begin{bmatrix} \dfrac{3k - \sqrt{3}\delta k_1}{2m} & 0 \\ 0 & \dfrac{3k + \sqrt{3}\delta k_1}{2m} \end{bmatrix}$$

The results obtained by applying sensitivity analysis techniques are again identical to those obtained by the dynamic analysis method.

V. Conclusions

The design sensitivity analysis of a repeated frequency is derived in the vibration mode space. Characteristics of the sensitivity analysis are revealed and demonstrated mathematically. It turns out that the sensitivity computation of a repeated frequency is irrelevant to the choice of the basis modes. Furthermore, we endow the sensitivities of a repeated frequency with a physical explanation: The sensitivities of a repeated frequency are extrema of the directional derivative in the related mode space. In addition, the effects of a design parameter variation on repeated frequencies are categorized, and a sufficient condition is presented to identify the maximum value of the lowest frequency when it is increased.

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C. Pierre
Associate Editor